Homological Algebra Seminar Week 3

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1 Chapter 1: Chain Complexes

We present below the formalism of chain complexes in a general abelian category, similar to the usual theory of chain complexes for R-modules. We fix throughout an abelian category $\mathcal{A}$.

Definition 1.1. A chain complex $C_\bullet$ in $\mathcal{A}$ is a family of objects $\{C_n\}_{n\in\mathbb{Z}}$ equipped with morphisms $d_n : C_n \to C_{n-1}$ (called differentials) such that $d_n \circ d_{n+1} = 0 \ \forall n \in \mathbb{Z}$.
We define the n-cycles and n-boundaries respectively by $Z_n(C_\bullet) = \ker(d_n)$, $B_n(C_\bullet) = \mathrm{im}(d_{n+1})$.

By abuse of notation we will often write the d_n in the above definition as d, so that the condition $d_n \circ d_{n+1} = 0$ becomes $d^2 = 0$.

Definition 1.2. We say $C_\bullet$ is bounded above if $\exists N \in \mathbb{Z}$ such that $C_n = 0$ $\forall n > N$. One similarly defines boundedness below. If $C_\bullet$ is bounded below by a and above by b, we say it has amplitude in $[a, b]$

Definition 1.3. A morphism of chain complexes $u_\bullet : C_\bullet, D_\bullet$ is a collection of morphisms $\{u_n : C_n \to D_n\}_{n\in\mathbb{Z}}$ satisfying appropriate compatibility conditions, namely such that the following diagram commutes $\forall n \in \mathbb{Z}$

$$\begin{tikzcd} C_n \arrow[r, "d_n"] \arrow[d, "u_n"'] & C_{n-1} \arrow[d, "u_{n-1}"] \\ D_n \arrow[r, "\delta_n"'] & D_{n-1} \end{tikzcd}$$

here we have noted the differentials of $C_\bullet$ by d, and those of $D_\bullet$ by δ.

One can easily verify that this defines a category of chain complexes over $\mathcal{A}$, which we denote by $\mathrm{Ch}(\mathcal{A})$.

1.1 Homology

Having defined chain complexes, the next natural step is to define the most important operation on these: homology functors.

Given a chain complex $C_\bullet$ with differentials d, note that $d_{n+1} : C_{n+1} \to C_n$ factors as $C_{n+1} \xrightarrow{e} \mathrm{im}(d_{n+1}) \xrightarrow{m} C_n$ for e, m epi and monic respectively. We thus obtain

$$0 = d_n \circ d_{n+1} = d_n \circ m \circ e \implies d_n \circ m = 0$$

By the universal property of kernels, m thus factors through $\ker(d_n)$.

Definition 1.4. Let $C_\bullet \in \mathrm{Ch}(\mathcal{A})$. The n-th homology of $C_\bullet$ is

$$H_n(C_\bullet) = \mathrm{coker}(\mathrm{im}(d_{n+1}) \to \ker(d_n))$$

with the implicit morphism a factor of m as above.

Remark 1.5. $H_n(C_\bullet) = 0$ iff C is exact at n (this is easy for R-modules, and one can then use Freyd-Mitchell).

We have yet to show that homology defines a collection of functors.

Proposition 1.6. *$H_n : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$ is a functor*

Proof. Consider a morphism $u_\bullet : C_\bullet \to D_\bullet$. Denote by d, δ the differentials of $C_\bullet, D_\bullet$ respectively. Let ι_n denote the natural morphisms $\iota_n : \ker(d_n) \to D_n$ $\forall n \in \mathbb{Z}$. Now we have $\forall n \in \mathbb{Z}$, $\delta_n \circ u_n \circ \iota_n = u_{n-1} \circ d_n \circ \iota_n = 0$ so by the universal property of kernels we get a diagram

$$\begin{array}{ccc}
\ker(d_n) & \longrightarrow & \ker(\delta_n) \\
\big\downarrow & & \big\downarrow \\
H_n(C_\bullet) & & H_n(D_\bullet)
\end{array}$$

In R-**mod**, the top map is $u_n|_{\ker(d_n)}$ and the diagram can be completed with a unique induced morphism $\theta_n : H_n(C_\bullet) \to H_n(D_\bullet)$ down below making the diagram commute. By Freyd-Mitchell we obtain such a morphism in the general case. If we thus define $H_n(C_\bullet)$ on morphisms by $H_n(u_\bullet) = \theta_n$, it is then easy to verify functoriality by the uniqueness property of θ_n. $\square$

Since we will usually be interested in maps preserving homology, it is worth defining a new notion of isomorphism.

Definition 1.7. A quasi-isomorphism of chain complexes over $\mathcal{A}$ is a morphism of chain complexes over $\mathcal{A}$, $u_\bullet : C_\bullet \to D_\bullet$, such that $H_n(u_\bullet)$ is an isomorphism $\forall n \in \mathbb{Z}$

Example 1.8.
1. An isomorphism is clearly a quasi-isomorphism
2. The converse is not true. A typical counterexample comes from the following morphism in $\mathrm{Ch}(\mathbb{Z}\text{-}\mathbf{mod})$

$$\begin{array}{ccccccc}
C_2 = 0 & \longrightarrow & C_1 = \mathbb{Z} & \longrightarrow & C_0 = \mathbb{Z} & \longrightarrow & C_{-1} = 0 \\
 & & \big\downarrow & & \big\downarrow & & \\
D_2 = 0 & \longrightarrow & D_1 = 0 & \longrightarrow & D_0 = \mathbb{Z}/2\mathbb{Z} & \longrightarrow & D_{-1} = 0
\end{array}$$

Here the map $C_0 \to D_0$ is just the usual quotient. One easily shows that this defines a quasi-isomorphism.

Having defined homology, the next natural step is to define cohomology.

Definition 1.9. A cochain complex in $\mathcal{A}$ is a family of objects $\{C^n\}_{n\in\mathbb{Z}}$ with a collection of morphisms $d^n : C^n \to C^{n+1}$ satisfying $d^{n+1} \circ d^n = 0\ \forall n \in \mathbb{Z}$. One defines cocycles, coboundaries, and cohomology in a dual way to all the definitions built on chain complexes.

1.2 $\mathrm{Ch}(\mathcal{A})$ as an abelian category

It is natural to expect that $\mathrm{Ch}(\mathcal{A})$ should have the structure of an abelian category. To show this let us first define kernels and cokernels appropriately.

Lemma 1.10. *In* $\mathrm{Ch}(\mathcal{A})$*, the kernel of* $f_\bullet : C_\bullet \to D_\bullet$ *is given by the chain complex* $\ker(f_\bullet)$ *with differentials induced by the universal properties of the kernels* $\ker(f_n)$ *(this will be elaborated on in the proof)*

Proof. Consider $f_\bullet : C_\bullet \to D_\bullet$ and the kernels $k_n : \ker(f_n) \to C_n$. Consider the following diagram

$$\begin{tikzcd}
 & & C_n \arrow[dd, "f_n"] \\
\ker(f_{n+1}) \arrow[r, dashed, "\exists! e_{n+1}"] \arrow[urr, bend left, "d_{n+1}\circ k_{n+1}"] \arrow[drr, bend right] & \ker(f_n) \arrow[ur] \arrow[dr, "0"] & \\
 & & D_n
\end{tikzcd}$$

where the bottom arrow is simply defined by the composition $f_n \circ d_{n+1} \circ k_{n+1}$, so as to make the diagram without e_{n+1} commute. Since

$$f_n \circ d_{n+1} \circ k_{n+1} = \delta_{n+1} \circ f_{n+1} \circ k_{n+1} = 0,$$

we are guaranteed the existence and uniqueness of e_{n+1} by the universal property of kernels. By the diagram, we have $k_n \circ e_{n+1} = d_n \circ k_{n+1}$ so $k_\bullet : \ker(f_\bullet) \to C_\bullet$ defines a morphism. It remains to show it satisfies the universal property of kernels.

Let let $K'_\bullet \in \mathrm{Ch}(\mathcal{A})$ and $j_\bullet : K'_\bullet \to C_\bullet$ be such that $f_\bullet \circ j_\bullet = 0$. At level n this gives $f_n \circ j_n = 0$ and thus by the universal property of kernels we have morphisms $u_n : K'_n \to \ker(f_n)$ with $k_n \circ u_n = j_n$. We just need to show these define a morphism of chain complexes. We have the diagram

$$
\begin{tikzcd}
K'_{n+1} \arrow[rrr, "g_{n+1}"] \arrow[dr, "u_{n+1}"] \arrow[dd, "j_{n+1}"'] & & & K'_n \arrow[dl, "u_n"'] \arrow[dd, "j_n"] \\
& \ker(f_{n+1}) \arrow[r, "e_{n+1}"] \arrow[dl, "k_{n+1}"'] & \ker(f_n) \arrow[dr, "k_n"] & \\
C_{n+1} \arrow[rrr, "d_{n+1}"] & & & C_n
\end{tikzcd}
$$

It commutes along the bottom trapezoid and the two triangles. We wish to show the top trapezoid commutes. We have

$$
\begin{aligned}
d_{n+1} \circ j_{n+1} = d_{n+1} \circ k_{n+1} \circ u_{n+1} &= k_n \circ e_{n+1} \circ u_{n+1} \\
= j_n \circ g_{n+1} &= k_n \circ u_n \circ g_{n+1}
\end{aligned}
$$

Since k_n is monic (it is a kernel), we get $e_{n+1} \circ u_{n+1} = u_n \circ g_{n+1}$ as required $\square$

Example 1.11. 1. If $\iota_\bullet : C_\bullet \to D_\bullet$ is injective (ie $\ker \iota_\bullet = 0$) we define

$$D_\bullet / C_\bullet = \operatorname{coker}(C_\bullet \to D_\bullet)$$

2. In R-**mod**, ie $C_\bullet \subset D_\bullet$ is a subcomplex, then D_n/C_n agrees with the usual quotient.

The next step in showing that $\mathrm{Ch}(\mathcal{A})$ is an abelian category is to show that it is additive and that it is Ab. We take as fact the following (it is easily shown):

Remark 1.12. In $\mathrm{Ch}(\mathcal{A})$ there is a zero object $\{0_\bullet : C_\bullet \to D_\bullet\}_{n \in \mathbb{Z}}$ and finite products $\{\prod_\alpha A_{\alpha,n}\}_{n \in \mathbb{Z}}$ with differentials

$$\prod_\alpha d_{\alpha,n} : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1}.$$

Additionally, each hom set has the structure of an abelian group given by $f_\bullet + g_\bullet = \{f_n + g_n\}_{n \in \mathbb{Z}}$ for $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ two morphisms, and inversion similarly componentwise defined.

Theorem 1.13. *The category* $\mathrm{Ch}(\mathcal{A})$ *is an abelian category*

Proof. We have already seen that $\mathrm{Ch}(\mathcal{A})$ is additive and Ab. We will show that that every monic is the ker of its coker. Showing every epic is the coker of its ker is similar.

Let $f_\bullet : B_\bullet \to C_\bullet$ be monic. We have a sequence $\ker(f_\bullet) \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{f_\bullet} C_\bullet$. By definition $f_\bullet \circ \iota_\bullet = 0$, so we obtain $\iota_\bullet = 0$. Thus $\iota_\bullet$ factors through the 0 map $0_\bullet \to B_\bullet$. We also have that the 0 map factors through $\iota_\bullet$, and thus by uniqueness of kernels we deduce that $\ker(f_\bullet) = 0_\bullet$. Now in abelian categories, $\ker(f) = 0 \iff f$ is monic. We deduce that f_n is monic $\forall n \in \mathbb{Z}$. Now let $q_\bullet : C_\bullet \to \operatorname{coker}(f_\bullet)$ be the natural morphism and $g_\bullet : D_\bullet \to C_\bullet$ be a morphism such that $q_\bullet \circ g_\bullet$ is zero. Then this also holds at level n and f_n is the ker of its coker so we get a diagram

$$\begin{tikzcd} B_n \arrow[r, "f_n"] & C_n \arrow[r, "q_n"] & \mathrm{coker}(f_n) \\ D_n \arrow[u, dashed, "\exists h_n"] \arrow[ur, "g_n"'] & & \end{tikzcd}$$

and h_n defines a morphism of chain complexes, so that $f_\bullet$ satisfies the universal property of $\ker(\mathrm{coker}(f_\bullet))$ as required. $\square$

1.3 Double complexes

The following objects will be of use when studying spectral sequences.

Definition 1.14. A double complex in $\mathcal{A}$ is a family $\{C_{p,q}\}_{(p,q)\in\mathbb{Z}^2}$ of objects in $\mathcal{A}$ with maps $d^h_{p,q} : C_{p,q} \to C_{p-1,q}$ and $d^v_{p,q} : C_{p,q} \to C_{p,q-1}$ satisfying

i) $d^h \circ d^h = 0$

ii) $d^v \circ d^v = 0$

iii) $d^v \circ d^h + d^h \circ d^v = 0$

Note that by the last condition, the squares in the following diagram usually don't commute

$$\begin{tikzcd} & \arrow[d] & \arrow[d] & \arrow[d] & \\ & C_{p-1,q+1} \arrow[l] \arrow[d, "d^v"] & C_{p,q+1} \arrow[l, "d^h"] \arrow[d, "d^v"] & C_{p+1,q+1} \arrow[l, "d^h"] \arrow[d, "d^v"] & \arrow[l] \\ & C_{p-1,q} \arrow[l] \arrow[d, "d^v"] & C_{p,q} \arrow[l, "d^h"] \arrow[d, "d^v"] & C_{p+1,q} \arrow[l, "d^h"] \arrow[d, "d^v"] & \arrow[l] \\ & C_{p-1,q-1} \arrow[l] \arrow[d] & C_{p,q-1} \arrow[l, "d^h"] \arrow[d] & C_{p+1,q-1} \arrow[l, "d^h"] \arrow[d] & \arrow[l] \\ & {} & {} & {} & \end{tikzcd}$$

We say the double complex is bounded if it only has finitely many non-zero objects

Remark 1.15. Although we usually write chain complexes from left to right, we have noted them from right to left here to label a double complex with $\mathbb{Z}^2$ in the obvious way

Remark 1.16. Although a double chain complex generally is not a chain complex of chain complexes, we would like to identify it with one. An object of the category $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$ is a complex $\to C_{\bullet,q} \to C_{\bullet,q-1} \to$. Given a double complex as above, define maps $f_{p,q} = (-1)^p d^v_{p,q}$: these define chain morphisms between the horizontal complexes, and it is easy to see that it defines an object of the category $\mathrm{Ch}(\mathrm{Ch}(\mathcal{A}))$.

One can build a chain complex from a double complex by "collapsing along the diagonal" in two different ways

Definition 1.17. Given a double complex $C_{\bullet,\bullet}$, its total complexes are the collections of objects

$$T_n^{\prod} = \prod_{p+q=n} C_{p,q}, \quad T_n^{\oplus} = \bigoplus_{p+q=n} C_{p,q}$$

Note this may not exist if $C_{\bullet,\bullet}$ is not bounded

We have not yet defined a chain complex as we need to specify differentials. We basically do this componentwise and apply the appropriate universal properties. Note that from a component $C_{p,q}$, there are two maps to objects of lower degree, d^h and d^v.

Proposition 1.18. *$T_\bullet^{\prod}$ and $T_\bullet^{\oplus}$, when they exist, naturally form chain complexes with differentials $d^h + d^v$*

Proof. We show how to properly construct these differentials for $T_\bullet^{\prod}$, the proof for $T_\bullet^{\oplus}$ is analogous (but using coproduct properties instead).
Let $n \in \mathbb{Z}$. For each pair $(p,q) \in \mathbb{Z}^2$ with $p+q=n$, we have a projection map $\pi_{p,q} : T_n^{\prod} \to C_{p,q}$. Composing with $d_{p,q}^h$ and $d_{p,q}^v$ gives us maps

$$\tilde{d}_{p,q}^h : T_n \to C_{p-1,q}, \quad \tilde{d}_{p,q}^v : T_n \to C_{p,q-1}$$

By the universal property of the product we thus get maps $d_n^h, d_n^v : T_n \to T_{n-1}$. We define the differential $\partial_n : T_n \to T_{n-1}$ by $\partial_n = d_n^h + d_n^v$ (addition taken in $\mathrm{Hom}_{\mathcal{A}}(T_n, T_{n-1})$). To check that this defines a chain complex, we have (since composition is bilinear in Ab categories)

$$\partial_{n-1} \circ \partial_n = d_{n-1}^h \circ d_n^h + d_{n-1}^v \circ d_n^h + d_{n-1}^h \circ d_n^v + d_{n-1}^v \circ d_n^v$$

One quickly verifies that $d_{n-1}^h \circ d_n^h$ is induced (through the universal property of the product) by the family $\{d_{p-1,q}^h \circ d_{p,q}^h \circ \pi_{p,q}\}_{p+q=n}$, that $d_{n-1}^v \circ d_n^v$ is induced by $\{d_{p,q-1}^v \circ d_{p,q}^v \circ \pi_{p,q}\}_{p+q=n}$, and that $d_{n-1}^v \circ d_n^h + d_{n-1}^h \circ d_n^v$ is induced by $\{(d_{p-1,q}^v \circ d_{p,q}^h + d_{p,q-1}^h \circ d_{p,q}^v) \circ \pi_{p,q}\}_{p+q=n}$. Since all of these are families of 0 objects by the definition of a double complex, we deduce that $\partial_{n-1} \circ \partial_n = 0$ as required. □