# Homological Algebra Seminar Week 3

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## 1 Chapter 1: Chain Complexes

We present below the formalism of chain complexes in a general abelian category, similar to the usual theory of chain complexes for R-modules. We fix throughout an abelian category  $\mathcal{A}$ .

**Definition 1.1.** A chain complex  $C_{\bullet}$  in  $\mathcal{A}$  is a family of objects  $\{C_n\}_{n\in\mathbb{Z}}$  equipped with morphisms  $d_n:C_n\to C_{n-1}$  (called differentials) such that  $d_n\circ d_{n+1}=0\ \forall n\in\mathbb{Z}$ .

We define the *n*-cycles and *n*-boundaries respectively by  $Z_n(C_{\bullet}) = \ker(d_n)$ ,  $B_n(C_{\bullet}) = \operatorname{im}(d_{n+1})$ .

By abuse of notation we will often write the  $d_n$  in the above definition as d, so that the condition  $d_n \circ d_{n+1} = 0$  becomes  $d^2 = 0$ .

**Definition 1.2.** We say  $C_{\bullet}$  is bounded above if  $\exists N \in \mathbb{Z}$  such that  $C_n = 0$   $\forall n > N$ . One similarly defines boundedness below. If  $C_{\bullet}$  is bounded below by a and above by b, we say it has amplitude in [a, b]

**Definition 1.3.** A morphism of chain complexes  $u_{\bullet}: C_{\bullet}, D_{\bullet}$  is a collection of morphisms  $\{u_n: C_n \to D_n\}_{n \in \mathbb{Z}}$  satisfying appropriate compatibility conditions, namely such that the following diagram commutes  $\forall n \in \mathbb{Z}$ 

$$C_n \xrightarrow{d_n} C_{n-1}$$

$$\downarrow u_n \downarrow \qquad \qquad \downarrow u_{n-1}$$

$$D_n \xrightarrow{\delta_n} D_{n-1}$$

here we have noted the differentials of  $C_{\bullet}$  by d, and those of  $D_{\bullet}$  by  $\delta$ .

One can easily verify that this defines a category of chain complexes over  $\mathcal{A}$ , which we denote by  $\mathrm{Ch}(\mathcal{A})$ .

#### 1.1 Homology

Having defined chain complexes, the next natural step is to define the most important operation on these: homology functors.

Given a chain complex  $C_{\bullet}$  with differentials d, note that  $d_{n+1}: C_{n+1} \to C_n$  factors as  $C_{n+1} \stackrel{e}{\longrightarrow} \operatorname{im}(d_{n+1}) \stackrel{m}{\longrightarrow} C_n$  for e, m epi and monic respectively. We thus obtain

$$0 = d_n \circ d_{n+1} = d_n \circ m \circ e \implies d_n \circ m = 0$$

By the universal property of kernels, m thus factors through  $\ker(d_n)$ .

**Definition 1.4.** Let  $C_{\bullet} \in Ch(\mathcal{A})$ . The *n*-th homology of  $C_{\bullet}$  is

$$H_n(C_{\bullet}) = \operatorname{coker}(\operatorname{im}(d_{n+1}) \to \ker(d_n))$$

with the implicit morphism a factor of m as above.

**Remark 1.5.**  $H_n(C_{\bullet}) = 0$  iff C is exact at n (this is easy for R-modules, and one can then use Freyd-Mitchell).

We have yet to show that homology defines a collection of functors.

**Proposition 1.6.**  $H_n: Ch(A) \to A$  is a functor

*Proof.* Consider a morphism  $u_{\bullet}: C_{\bullet} \to D_{\bullet}$ . Denote by  $d, \delta$  the differentials of  $C_{\bullet}, D_{\bullet}$  respectively. Let  $\iota_n$  denote the natural morphisms  $\iota_n: \ker(d_n) \to D_n$   $\forall n \in \mathbb{Z}$ . Now we have  $\forall n \in \mathbb{Z}, \ \delta_n \circ u_n \circ \iota_n = u_{n-1} \circ d_n \circ \iota_n = 0$  so by the universal property of kernels we get a diagram

$$\ker(d_n) \longrightarrow \ker(\delta_n) 
\downarrow 
H_n(C_{\bullet}) \qquad H_n(D_{\bullet})$$

In R-mod, the top map is  $u_n|_{\ker(d_n)}$  and the diagram can be completed with a unique induced morphism  $\theta_n: H_n(C_{\bullet}) \to H_n(D_{\bullet})$  down below making the diagram commute. By Freyd-Mitchell we obtain such a morphism in the general case. If we thus define  $H_n(C_{\bullet})$  on morphisms by  $H_n(u_{\bullet}) = \theta_n$ , it is then easy to verify functoriality by the uniqueness property of  $\theta_n$ .

Since we will usually be interested in maps preserving homology, it is worth defining a new notion of isomorphism.

**Definition 1.7.** A quasi-isomorphism of chain complexes over  $\mathcal{A}$  is a morphism of chain complexes over  $\mathcal{A}$ ,  $u_{\bullet}: C_{\bullet} \to D_{\bullet}$ , such that  $H_n(u_{\bullet})$  is an isomorphism  $\forall n \in \mathbb{Z}$ 

**Example 1.8.** 1. An isomorphism is clearly a quasi-isomorphism

2. The converse is not true. A typical counterexample comes from the following morphism in  $Ch(\mathbb{Z}\text{-}\mathbf{mod})$ 

$$C_2 = 0 \longrightarrow C_1 = \mathbb{Z} \longrightarrow C_0 = \mathbb{Z} \longrightarrow C_{-1} = 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$D_2 = 0 \longrightarrow D_1 = 0 \longrightarrow D_0 = \mathbb{Z}/2\mathbb{Z} \longrightarrow D_{-1} = 0$$

Here the map  $C_0 \to D_0$  is just the usual quotient. One easily shows that this defines a quasi-isomorphism.

Having defined homology, the next natural step is to define cohomology.

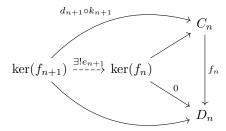
**Definition 1.9.** A cochain complex in  $\mathcal{A}$  is a family of objects  $\{C^n\}_{n\in\mathbb{Z}}$  with a collection of morphisms  $d^n:C^n\to C^{n+1}$  satisfying  $d^{n+1}\circ d^n=0\ \forall n\in\mathbb{Z}$ . One defines cocycles, coboundaries, and cohomology in a dual way to all the definitions built on chain complexes.

## 1.2 Ch(A) as an abelian category

It is natural to expect that Ch(A) should have the structure of an abelian category. To show this let us first define kernels and cokernels appropriately.

**Lemma 1.10.** In Ch(A), the kernel of  $f_{\bullet}: C_{\bullet} \to D_{\bullet}$  is given by the chain complex  $ker(f_{\bullet})$  with differentials induced by the universal properties of the kernels  $ker(f_n)$  (this will be elaborated on in the proof)

*Proof.* Consider  $f_{\bullet}: C_{\bullet} \to D_{\bullet}$  and the kernels  $k_n: \ker(f_n) \to C_n$ . Consider the following diagram

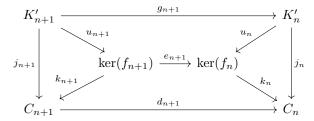


where the bottom arrow is simply defined by the composition  $f_n \circ d_{n+1} \circ k_{n+1}$ , so as to make the diagram without  $e_{n+1}$  commute. Since

$$f_n \circ d_{n+1} \circ k_{n+1} = \delta_{n+1} \circ f_{n+1} \circ k_{n+1} = 0,$$

we are guaranteed the existence and uniqueness of  $e_{n+1}$  by the universal property of kernels. By the diagram, we have  $k_n \circ e_{n+1} = d_n \circ k_{n+1}$  so  $k_{\bullet} : \ker(f_{\bullet}) \to C_{\bullet}$  defines a morphism. It remains to show it satisfies the universal property of kernels

Let let  $K'_{\bullet} \in \operatorname{Ch}(\mathcal{A})$  and  $j_{\bullet} : K'_{\bullet} \to C_{\bullet}$  be such that  $f_{\bullet} \circ j_{\bullet} = 0$ . At level n this gives  $f_n \circ j_n = 0$  and thus by the universal property of kernels we have morphisms  $u_n : K'_n \to \ker(f_n)$  with  $k_n \circ u_n = j_n$ . We just need to show these define a morphism of chain complexes. We have the diagram



It commutes along the bottom trapezoid and the two triangles. We wish to show the top trapezoid commutes. We have

$$d_{n+1} \circ j_{n+1} = d_{n+1} \circ k_{n+1} \circ u_{n+1} = k_n \circ e_{n+1} \circ u_{n+1}$$
$$= j_n \circ g_{n+1} = k_n \circ u_n \circ g_{n+1}$$

Since  $k_n$  is monic (it is a kernel), we get  $e_{n+1} \circ u_{n+1} = u_n \circ g_{n+1}$  as required  $\square$ 

**Example 1.11.** 1. If  $\iota_{\bullet}: C_{\bullet} \to D_{\bullet}$  is injective (ie ker  $\iota_{\bullet} = 0$ ) we define

$$D_{\bullet}/C_{\bullet} = \operatorname{coker}(C_{\bullet} \to D_{\bullet})$$

2. In R-mod, ie  $C_{\bullet} \subset D_{\bullet}$  is a subcomplex, then  $D_n/C_n$  agrees with the usual quotient.

The next step in showing that Ch(A) is an abelian category is to show that it is additive and that it is Ab. We take as fact the following (it is easily shown):

**Remark 1.12.** In Ch( $\mathcal{A}$ ) there is a zero object  $\{0_{\bullet}: C_{\bullet} \to D_{\bullet}\}_{n \in \mathbb{Z}}$  and finite products  $\{\prod_{\alpha} A_{\alpha,n}\}_{n \in \mathbb{Z}}$  with differentials

$$\prod_{\alpha} d_{\alpha,n} : \prod_{\alpha} A_{\alpha,n} \to \prod_{\alpha} A_{\alpha,n-1}.$$

Additionally, each hom set has the structure of an abelian group given by  $f_{\bullet} + g_{\bullet} = \{f_n + g_n\}_{n \in \mathbb{Z}}$  for  $f_{\bullet}, g_{\bullet} : C_{\bullet} \to D_{\bullet}$  two morphisms, and inversion similarly componentwise defined.

**Theorem 1.13.** The category Ch(A) is an abelian category

*Proof.* We have already seen that Ch(A) is additive and Ab. We will show that that every monic is the ker of its coker. Showing every epic is the coker of its ker is similar.

Let  $f_{\bullet}: B_{\bullet} \to C_{\bullet}$  be monic. We have a sequence  $\ker(f_{\bullet}) \xrightarrow{\iota_{\bullet}} B_{\bullet} \xrightarrow{f_{\bullet}} C_{\bullet}$ . By definition  $f_{\bullet} \circ \iota_{\bullet} = 0$ , so we obtain  $\iota_{\bullet} = 0$ . Thus  $\iota_{\bullet}$  factors through the 0 map  $0_{\bullet} \to B_{\bullet}$ . We also have that the 0 map factors through  $\iota_{\bullet}$ , and thus by uniqueness of kernels we deduce that  $\ker(f_{\bullet}) = 0_{\bullet}$ . Now in abelian categories,  $\ker(f) = 0 \iff f$  is monic. We deduce that  $f_n$  is monic  $\forall n \in \mathbb{Z}$ . Now let  $q_{\bullet}: C_{\bullet} \to \operatorname{coker}(f_{\bullet})$  be the natural morphism and  $g_{\bullet}: D_{\bullet} \to C_{\bullet}$  be a morphism such that  $q_{\bullet} \circ g_{\bullet}$  is zero. Then this also holds at level n and  $f_n$  is the ker of its coker so we get a diagram

$$B_n \xrightarrow{f_n} C_n \xrightarrow{q_n} \operatorname{coker}(f_n)$$

$$\exists h_n \mid g_n$$

$$D_n$$

and  $h_n$  defines a morphism of chain complexes, so that  $f_{\bullet}$  satisfies the universal property of  $\ker(\operatorname{coker}(f_{\bullet}))$  as required.

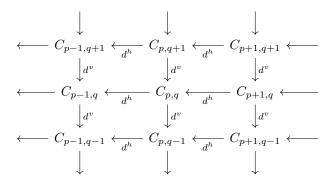
### 1.3 Double complexes

The following objects will be of use when studying spectral sequences.

**Definition 1.14.** A double complex in  $\mathcal{A}$  is a family  $\{C_{p,q}\}_{(p,q)\in\mathbb{Z}^2}$  of objects in  $\mathcal{A}$  with maps  $d_{p,q}^h:C_{p,q}\to C_{p-1,q}$  and  $d_{p,q}^v:C_{p,q}\to C_{p,q-1}$  satisfying

- i)  $d^h \circ d^h = 0$
- ii)  $d^v \circ d^v = 0$
- iii)  $d^v \circ d^h + d^h \circ d^v = 0$

Note that by the last condition, the squares in the following diagram usually don't commute



We say the double complex is bounded if it only has finitely many non-zero objects

**Remark 1.15.** Although we usually write chain complexes from left to right, we have noted them from right to left here to label a double complex with  $\mathbb{Z}^2$  in the obvious way

**Remark 1.16.** Although a double chain complex generally is not a chain complex of chain complexes, we would like to identify it with one. An object of the category Ch(Ch(A)) is a complex  $\to C_{\bullet,q} \to C_{\bullet,q-1} \to$ . Given a double complex as above, define maps  $f_{p,q} = (-1)^p d_{p,q}^v$ : these define chain morphisms between the horizontal complexes, and it is easy to see that it defines an object of the category Ch(Ch(A)).

One can build a chain complex from a double complex by "collapsing along the diagonal" in two different ways

**Definition 1.17.** Given a double complex  $C_{\bullet,\bullet}$ , its total complexes are the collections of objects

$$T_n^{\prod} = \prod_{p+q=n} C_{p,q}, \quad T_n^{\oplus} = \bigoplus_{p+q=n} C_{p,q}$$

Note this may not exist if  $C_{\bullet,\bullet}$  is not bounded

We have not yet defined a chain complex as we need to specify differentials. We basically do this componentwise and apply the appropriate universal properties. Note that from a component  $C_{p,q}$ , there are two maps to objects of lower degree,  $d^h$  and  $d^v$ .

**Proposition 1.18.**  $T^{\prod}_{\bullet}$  and  $T^{\oplus}_{\bullet}$ , when they exist, naturally form chain complexes with differentials  $d^h + d^v$ 

*Proof.* We show how to properly construct these differentials for  $T^{\Pi}_{\bullet}$ , the proof for  $T^{\oplus}_{\bullet}$  is analogous (but using coproduct properties instead). Let  $n \in \mathbb{Z}$ . For each pair  $(p,q) \in \mathbb{Z}^2$  with p+q=n, we have a projection map

 $\pi_{p,q}: T_n^{\prod} \to C_{p,q}$ . Composing with  $d_{p,q}^h$  and  $d_{p,q}^v$  gives us maps

$$\tilde{d}_{p,q}^h: T_n \to C_{p-1,q}, \quad \tilde{d}_{p,q}^v: T_n \to C_{p,q-1}$$

By the universal property of the product we thus get maps  $d_n^h, d_n^v: T_n \to T_{n-1}$ . We define the differential  $\partial_n: T_n \to T_{n-1}$  by  $\partial_n = d_n^h + d_n^v$  (addition taken in  $\operatorname{Hom}_{\mathcal{A}}(T_n, T_{n-1})$ ). To check that this defines a chain complex, we have (since composition is bilinear in Ab categories)

$$\partial_{n-1}\circ\partial_n=d^h_{n-1}\circ d^h_n+d^v_{n-1}\circ d^h_n+d^h_{n-1}\circ d^v_n+d^v_{n-1}\circ d^v_n$$

One quickly verifies that  $d_{n-1}^h \circ d_n^h$  is induced (through the universal property of the product) by the family  $\{d_{p-1,q}^h \circ d_{p,q}^h \circ \pi_{p,q}\}_{p+q=n}$ , that  $d_{n-1}^v \circ d_n^v$  is induced by  $\{d_{p,q-1}^v \circ d_{p,q}^v \circ \pi_{p,q}\}_{p+q=n}$ , and that  $d_{n-1}^v \circ d_n^h + d_{n-1}^h \circ d_n^v$  is induced by  $\{(d_{p-1,q}^v \circ d_{p,q}^h + d_{p,q-1}^h \circ d_{p,q}^v) \circ \pi_{p,q}\}_{p+q=n}$ . Since all of these are families of 0 objects by the definition of a double complex, we deduce that  $\partial_{n-1} \circ \partial_n = 0$  as required.